

Lower bounds for transition probabilities on graphs

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February 3, 2008

Abstract

The paper presents two results. The first one provides separate conditions for the upper and lower estimate of the distribution of the exit time from balls of a random walk on a weighted graph. The main result of the paper is that the lower estimate follows from the elliptic Harnack inequality. The second result is an off-diagonal lower bound for the transition probability of the random walk.

1 Introduction

Today a large amount of work is devoted to upper and two-sided estimates of heat kernels in different spaces (c.f. [7],[9],[10],[14],[18]). The main challenge is to find a connection between structural properties of the space and the behavior of the heat kernel. The study of the heat kernel in \mathbb{R}^n of course dates back to much earlier results among others to Moser [16],[17] and Aronson [1]. In these celebrated works chaining arguments were used. Chaining arguments appear in recent works as well. The present paper would like to provide a new one which replaces Aronson's chaining argument for graphs to obtain heat kernel lower estimates. The new approach eliminates the condition on the volume growth.

It is generally believed that the majority of the essential phenomena and difficulties related to diffusion are present in the discrete case. All that follows is in the discrete graph settings and discrete time, but one can see that most of the arguments carry over to the continuous case.

In the course of the study of the pre-Sierpinski gasket (c.f.[15] [2] and bibliography there) and other fractal structures upper or two-sided heat kernel estimates were given, which in the simplest case has the form as follows:

$$p_n(x, y) + p_{n+1}(x, y) \geq \frac{c}{V(x, n^{1/\beta})} \exp \left(-C \left(\frac{d^\beta(x, y)}{n} \right)^{\frac{1}{\beta-1}} \right) \quad (1)$$

$$p_n(x, y) \leq \frac{C}{V(x, n^{1/\beta})} \exp \left(-c \left(\frac{d^\beta(x, y)}{n} \right)^{\frac{1}{\beta-1}} \right) \quad (2)$$

In [13] necessary and sufficient conditions were given for (1) and (2). The standard route to the lower estimate typically goes via the diagonal upper and lower bound (and uses (3)). The present paper develops a different approach, which uses fewer assumptions. Neither volume growth conditions nor heat kernel upper estimates are used. Let us mention here that in [8] such estimates are given for strongly recurrent graphs without explicitly assuming the elliptic Harnack inequality. Meanwhile it is easy to show that the elliptic Harnack inequality follows directly from the conditions there.

During the proof of the upper estimate an interesting side-result can be observed. The distribution of the exit time from a ball has an upper estimate under a particular condition. Consider T_B , the exit time from a ball $B = B(x, R)$. The expected value of T_B is denoted by $E(x, R) = E(T_B | X_0 = x)$ assuming that the starting point is x . On many fractals (or fractal type graph) the space-time scaling function is R^β , $cR^\beta \leq E(x, R) \leq CR^\beta$, for $\beta \geq 2$, $C > 1 > c > 0$ constants and this property implies that

$$\mathbb{P}(T_B < n | X_0 = x) \leq C \exp \left(-c \left(\frac{R^\beta}{n} \right)^{\frac{1}{\beta-1}} \right). \quad (3)$$

This estimate (and the lower counterpart as well in the case of the Brownian motion on the Sierpinski gasket) was given first in [5] and later an independent proof was provided for more general settings in [12] using also a chaining argument.

One might wonder about the condition which ensures the same (up to the constants) lower bound.

The main results are illustrated for the particular case $cR^\beta \leq E(x, R) \leq CR^\beta$ postponing the general statements after the necessary definitions. If the elliptic Harnack inequality (see Definition (20)) holds, then for $n \geq R$, $B = B(x, R)$

$$\mathbb{P}(T_B < n | X_0 = x) \geq c \exp \left(-C \left(\frac{R^\beta}{n} \right)^{\frac{1}{\beta-1}} \right). \quad (4)$$

and

$$p_n(x, y) + p_{n+1}(x, y) \geq \frac{c}{V(x, n^{1/\beta})r^D} \exp \left(-C \left(\frac{d^\beta(x, y)}{n} \right)^{\frac{1}{\beta-1}} \right), \quad (5)$$

where $r = \left(\frac{n}{d(x, y)} \right)^{\frac{1}{\beta-1}}$, $n \geq d(x, y) \geq 0$, $n > 0$ and D is a fixed constant.

The results are new from several points of view. First of all, to our best knowledge, lower estimates like (4) are new in this generality. One should also observe that the lower estimate (4) matches with the upper one (3) obtained from stronger assumptions. The key steps are given in Proposition 27 and 30 which help to control the probability to hit a nearby ball, which is usually more difficult than to control exit from a ball.

In Section 2 the necessary definitions are introduced. In Section 3 we give the general form and proof of (4). In Section 4 we show a heat kernel lower bound (better than (5)) for very strongly recurrent walks and in Section 5 we show a result which contains (5) as a particular case.

Acknowledgement

The author is indebted to Professor Alexander Grigor'yan for many useful discussions and particularly for the remarks helped to clarify the proof of Lemma 26. Thanks are due to the referee for the careful reading of the paper and many helpful suggestions.

2 Basic definitions

In this section we give the basic definitions for our discussion. Let us consider an infinite connected graph Γ . We assume, for sake of simplicity, that there are no multiple edges and loops.

Let $\mu_{x,y} = \mu_{y,x} > 0$ be a symmetric weight function given on the edges $x \sim y$. These weights induce a measure $\mu(x)$

$$\begin{aligned} \mu(x) &= \sum_{y \sim x} \mu_{x,y}, \\ \mu(A) &= \sum_{y \in A} \mu(y) \end{aligned}$$

on the vertex sets $A \subset \Gamma$. The weights $\mu_{x,y}$ define a reversible Markov chain $X_n \in \Gamma$, i.e., a random walk on the weighted graph (Γ, μ) with transition probabilities

$$\begin{aligned} P(x, y) &= \frac{\mu_{x,y}}{\mu(x)}, \\ P_n(x, y) &= \mathbb{P}(X_n = y | X_0 = x). \end{aligned}$$

The transition "density" or heat kernel for the discrete random walk is defined as

$$p_n(x, y) = \frac{1}{\mu(y)} P_n(x, y).$$

To avoid parity problems we introduce

$$\tilde{p}_n(x, y) = p_n(x, y) + p_{n+1}(x, y).$$

We will assume in the whole paper that the one step transition probabilities are uniformly separated from zero, i.e. there is a $p_0 > 0$ such that

$$P(x, y) \geq p_0 > 0 \tag{p_0}$$

for all $x \sim y$, $x, y \in \Gamma$.

Definition 1 *The graph is equipped with the usual (shortest path length) graph distance $d(x, y)$ and open metric balls are defined for $x \in \Gamma$, $R > 0$ as*

$$\begin{aligned} B(x, R) &= \{y \in \Gamma : d(x, y) < R\}, \\ S(x, R) &= \{y \in \Gamma : d(x, y) = R\} \end{aligned}$$

and the μ -measure of $B(x, R)$ denoted by $V(x, R)$

$$V(x, R) = \mu(B(x, R)).$$

Definition 2 *We use*

$$\overline{A} = \{y \in \Gamma : \exists x \in A, x \sim y\}$$

for the closure of a set A , Denote $\partial A = \overline{A} \setminus A$ and $A^c = \Gamma \setminus A$ the complement of A .

Definition 3 *In general, $a_\xi \simeq b_\xi$ will mean that there is a $C > 0$ such that for all ξ*

$$\frac{1}{C} a_\xi \leq b_\xi \leq C a_\xi.$$

Unimportant constants will be denoted by c, C and they may change from place to place absorbing other intermediate constants.

Let us introduce the exit time T_A for a set $A \subset \Gamma$.

Definition 4 *The exit time from a set A is defined as*

$$T_A = \inf\{t \geq 0 : X_t \in A^c\},$$

its expected value is denoted by

$$\begin{aligned} E_y(A) &= \mathbb{E}(T_A | X_0 = y), \\ E_y(x, R) &= E_y(B(x, R)) \end{aligned}$$

and we will use the $E = E(x, R) = E_x(B(x, R))$ and $T_{x,R} = T_{B(x,R)}$ short notations.

The definition implies that

$$E(x, 1) = 1. \quad (6)$$

Definition 5 *The hitting time τ_A of a set $A \subset \Gamma$ is defined by*

$$\tau_A = T_{A^c},$$

and we write $\tau_{x,R} = \tau_{B(x,R)}$.

Definition 6 *We introduce the maximal exit time for $x \in \Gamma, R > 0$ by*

$$\overline{E}(x, R) = \max_{y \in B(x, R)} E_y(x, R).$$

Definition 7 *One of the key assumptions in our study is the condition (\overline{E}) : there is a $C > 0$ such that for all $x \in \Gamma, R > 0$*

$$\overline{E}(x, R) \leq CE(x, R) \quad (7)$$

is true.

Definition 8 *We say that the time comparison principle holds for (Γ, μ) if there is a $C_T > 1$ constant such that for any $x \in \Gamma, R > 0, y \in B(x, R)$*

$$\frac{E(y, 2R)}{E(x, R)} \leq C_T. \quad (8)$$

Proposition 9 *From the time comparison principle it follows that*

$$\frac{E(x, 2R)}{E(x, R)} \leq C_T, \quad (9)$$

$$\overline{E}(x, R) \leq CE(x, R) \quad (10)$$

and there is a constant A_T such that for all $x \in \Gamma, R > 0$

$$E(x, A_T R) \geq 2E(x, R). \quad (11)$$

Remark 10 For the easy proofs see [19]. One can deduce that (9) is equivalent to that there is a $\beta \geq 1$ and $C > 0$ such that for all $R > r > 0, x \in \Gamma, y \in B(x, R)$

$$\frac{E(x, 2R)}{E(x, r)} \leq C \left(\frac{2R}{r} \right)^\beta, \quad (12)$$

and it implies

$$E(x, R) \leq CR^\beta.$$

Similarly (11) is equivalent to that there are $\beta' > 0, c > 0$ such that for all $x \in \Gamma, R > r > 0, y \in B(x, R)$

$$c \left(\frac{2R}{r} \right)^{\beta'} \leq \frac{E(x, 2R)}{E(x, r)} \quad (13)$$

and from (12) it follows that

$$E(x, R) \geq cR^{\beta'}.$$

Remark 11 It is also easy to see that (\overline{E}) implies (11) and hence (13) as well.

Definition 12 For the mean exit time $E(x, R)$, $R \in \mathbb{N}$ we define the inverse in the second variable

$$e(x, n) = \min \{r \in \mathbb{N} : E(x, r) \geq n\}.$$

Remark 13 The inverse function $e(x, n)$ is well-defined since $E(x, R)$ is strictly increasing for $R \in \mathbb{N}$ (cf. [20]).

Definition 14 For a given $x \in \Gamma, n \geq R > 0$ let us define $k = k(x, n, R)$ as the maximal integer for which

$$\frac{n}{k} \leq q \min_{z \in B(x, R)} E\left(z, \frac{R}{k}\right),$$

where q is a fixed constant. Let $k = 1$ by definition if there is no such integer.

Definition 15 Let us denote by $\pi_{x,y}$ the the union of the vertices of shortest paths connecting x and y .

Definition 16 For $x, y \in \Gamma, n \geq R > 0, C > 0$ let us define $l = l_C(x, y, n, R)$ as the minimal integer for which

$$\frac{n}{l} \geq Q \max_{z \in \pi_{x,y}} E\left(z, \frac{CR}{l}\right),$$

where Q is a fixed constant (to be specified later.). Let $l = R$ by definition if there is no such integer. If $d(x, y) = R$ we will use the shorter notation $l_C(x, y, n) = l_C(x, y, n, d(x, y))$

Definition 17 For a given $x \in \Gamma, n \geq R > 0$ let us define

$$\nu = \nu(x, n, R) = \min_{y \in S(x, 2R)} l_9(x, y, n, R).$$

Remark 18 One can show easily from (12) that

$$k(x, n, R) \geq c \left(\frac{E(x, R)}{n} \right)^{\frac{1}{\beta-1}}$$

and similarly using (13) that if $\beta' > 1$ that

$$\nu(x, n, R) \leq C \left(\frac{E(x, R)}{n} \right)^{\frac{1}{\beta'-1}}.$$

Definition 19 A function $h : \Gamma \rightarrow \mathbb{R}$ said to be harmonic on $A \subset \Gamma$ if it is defined on \overline{A} and

$$\sum_{y \in \Gamma} P(x, y) h(y) = h(x) \quad \text{for all } x \in A.$$

Definition 20 We say that the weighted graph (Γ, μ) satisfies (H) the elliptic Harnack inequality if there is a constant $C > 0$ such that for all $x \in \Gamma, R > 0$ and for any non-negative harmonic function u which is harmonic on $B(x, 2R)$, the following inequality holds

$$\max_{B(x, R)} u \leq C \min_{B(x, R)} u.$$

If the weights of the edges are considered as wires, the whole graph can be seen as an electric network. Resistances are defined using the usual capacity notion.

Definition 21 On (Γ, μ) the Dirichlet form is defined as

$$\mathcal{E}(f, f) = \sum_{y \sim z} \mu_{y,z} (f(y) - f(z))^2$$

and the inner product is

$$(f, f) = \sum_y f^2(x) \mu(x).$$

Definition 22 For any disjoint sets A, B the capacity is defined via the Dirichlet form \mathcal{E} by

$$\text{cap}(A, B) = \inf \{ \mathcal{E}(f, f) : f|_A = 1, f|_B = 0 \}.$$

The resistance is defined then as

$$\rho(A, B) = \frac{1}{\text{cap}(A, B)}.$$

In particular we will use the following notations: for $R > r > 0, x \in \Gamma$

$$\rho(x, r, R) = \rho(B(x, r), B^c(x, R)).$$

3 Distribution of the exit time

In this section we prove the following theorem.

Theorem 23 Assume that the weighted graph (Γ, μ) satisfies (p_0) .

1. If (\overline{E}) holds, then there are $c, C > 0$ such that for all $n \geq R > 0, x \in \Gamma$

$$\mathbb{P}(T_{x,R} < n) \leq C \exp(-ck(x, n, R))$$

is true.

2. If (Γ, μ) satisfies the elliptic Harnack inequality (H) , then there are $c, C > 0$ such that for all $n \geq R > 0, x \in \Gamma$

$$\mathbb{P}(T_{x,R} < n) \geq c \exp(-C\nu(x, n, R)). \quad (14)$$

The proof of the upper bound was given in [19]. The lower bound is based on a new chaining argument. First we need some propositions.

Proposition 24 Assume that the weighted graph (Γ, μ) satisfies (p_0) and (\overline{E}) , then there is a $c > 0$ such that for all $x \in \Gamma, n, R > 0$

$$\mathbb{P}(T_{x,R} > n) > c,$$

if $n \leq \frac{1}{4}E(x, R)$.

Proof. From Lemma 5.3 of [19] one has for $A = B(x, R)$ that

$$\mathbb{P}(T_{x,R} \leq n) \leq 1 - \frac{E(x, R)}{2\overline{E}(x, R)} + \frac{n}{\overline{E}(x, R)}.$$

From the condition $\frac{\overline{E}(x, R)}{E(x, R)} \leq C$ and $n \leq \frac{1}{4}E(x, R)$ one obtains

$$\mathbb{P}(T_{x,R} > n) \geq \frac{E(x, R) - 2n}{2\overline{E}(x, R)} \geq \frac{1}{4C}.$$

■

Lemma 25 If (Γ, μ) satisfies (p_0) and the elliptic Harnack inequality (H) , then for $x \in \Gamma, r > 0, K > L \geq 1, B = B(x, Kr), S = \{y : d(x, y) = Lr\}$

$$\min_{w \in S} g^B(w, x) \simeq \rho(x, Lr, Kr) \simeq \max_{v \in S} g^B(v, x). \quad (15)$$

Proof. See Barlow's proof ([4], Proposition 2). ■

Lemma 26 If (Γ, μ) satisfies (p_0) and the elliptic Harnack inequality (H) , then there is a $c_1 > 0$ such that for all $x \in \Gamma, r > 0, w \in \overline{B}(x, 4r)$

$$\mathbb{P}_w(\tau_{x,r} < T_{x,5r}) > c_1. \quad (16)$$

Proof. The investigated probability

$$u(w) = \mathbb{P}_w(\tau_{x,r} < T_{x,5r}) \quad (17)$$

is the capacity potential between $\Gamma \setminus B(x, 5r)$ and $B(x, r)$ and clearly harmonic in $A = B(x, 5r) \setminus B(x, r)$. Write $B = B(x, 5r)$. So it can be as usual decomposed

$$u(w) = \sum_z g^B(w, z) \pi(z)$$

with the proper capacity measure $\pi(z)$ with support in $S(x, r)$, $\pi(A) = 1/\rho(x, r, 5r)$. From the maximum (minimum) principle it follows that the

minimum of $u(w)$ is attained on the boundary, $w \in S(x, 4r - 1)$ and from the Harnack inequality for $g^B(w, \cdot)$ in $B(x, 2r)$ that

$$\min_{z \in \overline{B}(x, r+1)} g^B(w, z) \geq c g^B(w, x),$$

$$u(w) = \sum_z g^B(w, z) \pi(z) \geq \frac{c g^B(w, x)}{\rho(x, r, 5r)}.$$

From Lemma 25 we know that

$$\max_{y \in B(x, 5r) \setminus B(x, 4r)} g^B(y, x) \simeq \rho(x, 4r, 5r) \simeq \min_{w \in B(x, 4r)} g^B(w, x).$$

which means that

$$u(w) \geq c \frac{\rho(x, 4r, 5r)}{\rho(x, r, 5r)}. \quad (18)$$

Similarly from Lemma 25 it follows that

$$\max_{v \in B(x, 5r) \setminus B(x, r)} g^B(v, x) \simeq \rho(x, r, 5r) \simeq \min_{w \in B(x, r)} g^B(w, x).$$

Finally if $y_0 \in \partial B(x, r)$ is on the ray from x to $y \in \partial B(x, 4r)$ then iterating the Harnack inequality along a finite chain of balls of radius $r/4$ along this ray from y_0 to y one obtains

$$g^B(y, x) \simeq g^B(y_0, x),$$

which results that

$$\rho(x, 4r, 5r) \geq c \rho(x, r, 5r),$$

and the statement follows from (18). ■

Proposition 27 *Assume that the weighted graph (Γ, μ) satisfies (p_0) and (H) . Then there are $c_0, c_1 > 0$ such that for all $x, z \in \Gamma, r > 0, d(x, z) \leq 4r, m > \frac{2}{c_1} E(x, 9r)$*

$$\mathbb{P}_x(\tau_{z,r} < m) > c_0.$$

Proof. We start with the following simple estimate:

$$\begin{aligned} \mathbb{P}_x(\tau_{z,r} < m) &\geq \mathbb{P}_x(\tau_{z,r} < T_{x,9r} < m) \\ &= \mathbb{P}_x(\tau_{z,r} < T_{x,9r}) - \mathbb{P}_x(\tau_{z,r} < T_{x,9r}, T_{x,9r} \geq m) \\ &\geq \mathbb{P}_x(\tau_{z,r} < T_{x,9r}) - \mathbb{P}_x(T_{x,9r} \geq m). \end{aligned}$$

On one hand

$$\mathbb{P}_x(T_{x,9r} \geq m) \leq \frac{E(x, 9r)}{m} \leq \frac{E(x, 9r)}{\frac{2}{c_1} E(x, 9r)} < c_1/2$$

and on the other hand $B(z, 5r) \subset B(x, 9r)$, hence

$$\mathbb{P}_x(\tau_{z,r} < T_{x,9r}) \geq \mathbb{P}_x(\tau_{z,r} < T_{z,5r}),$$

and Lemma 26 can be applied to get

$$\mathbb{P}_x(\tau_{z,r} < T_{z,5r}) \geq c_1.$$

The result follows with $c_0 = c_1/2$. ■

Lemma 28 *Let us assume that $x \in \Gamma, m, r, l \geq 1, 0 \leq u \leq 3l - 2, r = (3l - 2)r - u, y \in S(x, r + r)$ and write $n = ml$, then*

$$\mathbb{P}_x(\tau_{y,r} < n) \geq \min_{w \in \pi_{x,y}, 2r-3 \leq d(z,w) \leq 4r} \mathbb{P}_z^l(\tau_{w,r} < m).$$

where $\pi_{x,y}$ is the union of vertices of all possible shortest paths from x to y .

Remark 29 *The statement (and its consequences) can be sharpened if we consider separately all possible paths of comparable length to the shortest one and consider the minimum over the vertices of each path than the maximum for the paths. We omit this refinement here.*

Proof. We define a chain of balls. For $1 \leq l \leq d(x, y) - r$ let us consider a sequence of vertices $x_0 = x, x_1, \dots, x_l = y, x_i \in \pi_{x,y}$ in the following way: $d(x_{i-1}, x_i) = r - \delta_i$, where $\delta_i \in \{0, 1, 2, 3\}$ for $i = 1 \dots l$ and

$$u = \sum_{i=1}^l \delta_i$$

$$R = (3l - 2)r - \sum_{i=1}^l \delta_i = (3l - 2)r - u.$$

Let $\tau_i = \tau_{x_i, r}$ and $s_i = \tau_i - \tau_{i-1}, A_i = \{s_i < m\}, \mathbb{A}_i = \cap_{j=1}^i A_j$ for $i = 1, \dots, l, \tau_0 = 0$. Let us use the notation $D_i(z_i) = A_i \cap \{X_{\tau_i} = z_i\}$. One can observe that $\cap_{i=1}^l A_i$ means that the walk takes less than m steps between the first hit of the consecutive $B_i = B(x_i, r)$ balls, consequently

$$\mathbb{P}_x(\tau_{y,r} < n) \geq \mathbb{P}_x(\mathbb{A}_l)$$

We also note that $s_i = \min \{k : X_k \in B_i | X_0 \in \partial B_{i-1}\}$. From this one obtains the following estimates denoting $z_0 = x$

$$\begin{aligned} \mathbb{P}_x(\tau_{y,r} < n) &\geq \mathbb{P}_x(\mathbb{A}_l) \\ &= \sum_{z_{l-1} \in \partial B_{l-1}} \mathbb{P}_x[\mathbb{A}_{l-2} \cap D_{l-1}(z_{l-1}) \cap A_l] \end{aligned}$$

Now we use the Markov property.

$$\begin{aligned} &\sum_{z_{l-1} \in \partial B_{l-1}} \mathbb{P}_x[\mathbb{A}_{l-2} \cap D_{l-1}(z_{l-1}) \cap A_l] \\ &= \sum_{z_{l-1} \in \partial B_{l-1}} \mathbb{P}_x[A_l | \mathbb{A}_{l-2} \cap D_{l-1}(z_{l-1})] \mathbb{P}_x[\mathbb{A}_{l-2} \cap D_{l-1}(z_{l-1})] \\ &= \sum_{z_{l-1} \in \partial B_{l-1}} \mathbb{P}_{z_{l-1}}(s_l < m) \mathbb{P}_x(\mathbb{A}_{l-2} \cap D_{l-1}(z_{l-1})) \\ &\geq \min_{w \in \pi_{x,y}, 2r-3 \leq d(z,w) \leq 4r} \mathbb{P}_z(\tau_{w,r} < m) \mathbb{P}(\mathbb{A}_{l-1}), \end{aligned}$$

Denoting $q = \min_{w \in \pi_{x,y}, 2r-3 \leq d(z,w) \leq 4r} \mathbb{P}_z(\tau_{w,r} < m)$ we have

$$\mathbb{P}(\mathbb{A}_l) \geq q \mathbb{P}(\mathbb{A}_{l-1})$$

then iterating this expression gives the result. ■

Now we can prove the main ingredient of this section, which helps to control the probability of hitting a nearby ball.

Proposition 30 *Assume that the weighted graph (Γ, μ) satisfies (p_0) and the elliptic Harnack inequality (H) . Then there are $c, C, C' > 0$ such that for all $x, y \in \Gamma, r \geq 1, n > d(x, y) - r, d(x, y) \leq 4r$*

$$\mathbb{P}_x(\tau_{y,r} < n) \geq c \exp[-C' l_C(x, y, n, d(x, y) - r)].$$

Proof. If $n > \frac{2}{c_1} E(x, 9R)$, then the statement follows from Proposition 27. Also if $r \leq 9$, then $\frac{R}{3r} \leq l \leq R$, so from (p_0) the trivial lower estimate

$$\mathbb{P}_x(\tau_{y,r} < n) \geq c \exp\left(-27 \left(\log \frac{1}{p_0}\right) l\right)$$

gives the statement. If $n < \frac{2}{c_1} E(x, 9R)$ and $r \geq 10$, then $l_9(x, y, n, R) > 1$ and $R = (3l - 2)r - u \geq 34$. Let us use Proposition 27 and Lemma 28. The latter one states that

$$\mathbb{P}_x(\tau_{y,r} < n) \geq \min_{w \in \pi_{x,y}, 2r-3 \leq d(z,w) \leq 4r} \mathbb{P}_z^l(\tau_{w,r} < m). \quad (19)$$

Consider the following straightforward estimates for $r \geq 10, R \geq 10$.

$$\begin{aligned} 9r &\leq 10(r-1) \leq 10 \left(\frac{R+u}{3l-2} - 1 \right) \leq 10 \left(\frac{R+3l}{3l-2} - 1 \right) = 10 \frac{R+2}{3l-2} \\ &\leq \frac{4R}{(l-1)} \leq 8 \frac{R}{l} < 9 \frac{R}{l}. \end{aligned}$$

Let us also note $r = \frac{R+u}{3l-2} > \frac{R}{4}$ for all $l > 1$. If $l = l_9(x, y, n, R)$,

$$m = \frac{n}{l} > \frac{2}{c_1} E(w, 9r) = \frac{2}{c_1} E \left(w, 9 \frac{R+u}{3l-2} \right),$$

and $2r \leq d(z, w) \leq 4r$ then we can apply Proposition 27 to obtain the uniform lower estimate

$$\mathbb{P}_z^l(\tau_{w,r} < m) > c$$

for $w \in \pi_{x,y}$. This yields the uniform lower bound for all probabilities in (19). ■

Proof of Theorem 23. The upper estimate of Theorem 23 can be seen along the lines of the proof of Theorem 5.1 in [19]. The lower bound is immediate from Proposition 30 by using that

$$\mathbb{P}_x(T_{x,R} < n) \geq \mathbb{P}_x(\tau_{y,r} < n).$$

and minimizing $l_9(x, y, n)$ for $d = d(x, y) = 2R, y \in S(x, 2R), \frac{d}{4} = R/2 \leq r < R$. ■

4 Very strongly recurrent graphs

Definition 31 *Following [2] we say that a graph is very strongly recurrent (VSR) if there is a $c > 0$ such that for all $x \in \Gamma, r > 0, w \in \partial B(x, r)$*

$$\mathbb{P}_w(\tau_x < T_{x,2r}) \geq c.$$

In this section we deduce an off-diagonal heat kernel lower bound for very strongly recurrent graphs. The proof is based on Theorem 23 and the fact that very strong recurrence implies the elliptic Harnack inequality (c.f. [2]). Let us mention here that the strong recurrence was defined among others in [19] and one can see easily that strong recurrence in conjunction with the elliptic Harnack inequality is equivalent to very strong recurrence. It is worth to note, that the usually considered finitely ramified fractals and their pre-fractal graphs are (very) strongly recurrent.

Theorem 32 *Let us assume that (Γ, w) satisfies (p_0) and is very strongly recurrent furthermore satisfies (\overline{E}) . Then there are $c, C > 0$ such that for all $x, y \in \Gamma, n \geq d(x, y)$*

$$\tilde{p}_n(x, y) \geq \frac{c}{V(x, e(x, n))} \exp \left[-Cl_9 \left(x, y, \frac{1}{2}n, d \right) \right],$$

where $d = d(x, y)$.

Remark 33 *Typical examples for very strongly recurrent graphs are pre-fractal skeletons of p.c.f. self similar sets (for the definition, and further reading see [2] and [3]). We recall an example of a very strongly recurrent graph for which volume doubling does not hold but the elliptic Harnack inequality does. The example is due to Barlow (Lemma 5.1, 5.2 of [2]) and Delmotte's (c.f. [11] Section 5.). Let us consider Γ_1, Γ_2 two trees which are (VSR) and assume that $V_i(x, R) \simeq R^{\alpha_i}, E(x, R) \simeq R^{\beta_i}, \alpha_1 \neq \alpha_2$,*

$$\gamma = \beta_1 - \alpha_1 = \beta_2 - \alpha_2 > 0$$

which basically means that

$$\rho(x, R, 2R) \simeq R^\gamma$$

for both graphs. Such trees are constructed in [2]. Let Γ be the joint of Γ_1 and Γ_2 , which means that two vertices O_1, O_2 are chosen and identified (for details see [11]). One can also see that Γ is (VSR) and hence satisfies the Harnack inequality but not the volume doubling property. This means that Γ is an example for graphs that satisfies the Harnack inequality but not the usual volume properties.

It was realized some time ago that the so-called near diagonal lower estimate (20) is a crucial step to obtain off-diagonal lower estimates. Here we utilize the fact that the near diagonal lower bound is an easy consequence of very strong recurrence. As we shall see the proof does not use the diagonal upper estimate and assumption on the volume.

Proposition 34 *Assume (p_0) and (\overline{E}) , then there is a $c > 0$ such that for all $x \in \Gamma, n > 0$*

$$p_{2n}(x, x) \geq \frac{c}{V(x, e(x, 2n))}.$$

For the proof see Proposition 6.4 of [19].

Proposition 35 *Let us assume that (Γ, μ) satisfies (p_0) . If the graph is very strongly recurrent and (\overline{E}) holds, then there are $c, c' > 0$ such that for all $x, y \in \Gamma, m \geq \frac{2}{c'}E(x, 2d(x, y))$*

$$\tilde{p}_m(x, y) \geq \frac{c}{V(x, e(x, m))}. \quad (20)$$

Proof. The proof starts with a first hit decomposition and uses Proposition 34.

$$\begin{aligned} \tilde{p}_m(y, x) &\geq \sum_{i=0}^{m-1} \mathbb{P}_y(\tau_x = i) \tilde{p}_{m-i}(x, x) \geq \mathbb{P}_y(\tau_x < m) \tilde{p}_m(x, x) \\ &\geq \frac{c}{V(x, e(x, m))} \mathbb{P}_y(\tau_x < m). \end{aligned}$$

Denote $r = d(x, y)$,

$$\mathbb{P}_y(\tau_x < m) \geq \mathbb{P}_y(\tau_x < T_{x, 2r} < m) \geq \mathbb{P}_y(\tau_x < T_{x, 2r}) - \mathbb{P}_y(T_{x, 2r} \geq m).$$

From (VSR) we have that $\mathbb{P}_y(\tau_x < T_{x, 2r}) > c$ so from $m \geq \frac{2}{c'}E(x, 2r)$ and from the Markov inequality it follows that

$$\mathbb{P}_y(T_{x, 2r} \geq m) \leq \frac{E(x, 2r)}{m} \leq c'/2.$$

Consequently we have that $\mathbb{P}_y(\tau_x < m) > c'/2$ and the result follows. ■

Proof of Theorem 32. If $l = l_9(x, y, n, d(x, y)) = 1$, then $n >$

$\frac{2}{c'}E(x, 9d) > \frac{2}{c'}E(x, 2d)$ and the statement follows from Proposition 35. Let us assume that $l > 1$ and start with a path decomposition. Denote $m = \lfloor \frac{n}{l} \rfloor$, $r = \lfloor \frac{R}{l} \rfloor$, $S = \{y : d(x, y) = r\}$, $\tau = \tau_S$

$$\begin{aligned} \tilde{p}_n(y, x) &= \frac{1}{\mu(x)} \mathbb{P}_y(X_n = x \text{ or } X_{n+1} = x) \\ &\geq \sum_{i=0}^{n-m-1} \sum_{w \in S} \mathbb{P}_y(X_\tau = w, \tau = i) \min_{w \in S} \tilde{p}_{n-i}(w, x) \\ &\geq \sum_{i=0}^{n-m-1} \mathbb{P}_y(\tau = i) \min_{w \in S} \tilde{p}_{n-i}(w, x). \end{aligned}$$

The next step is to use the near diagonal lower estimate:

$$\begin{aligned}
\tilde{p}_n(y, x) &\geq \sum_{i=0}^{n-m-1} \mathbb{P}_y(\tau = i) \min_{w \in S} \tilde{p}_{n-i}(w, x) \\
&\geq \sum_{i=0}^{n-m-1} \mathbb{P}_y(\tau = i) \frac{c}{V(x, e(x, n-i))} \\
&\geq \mathbb{P}_y\left(\tau < \frac{n}{2}\right) \frac{c}{V(x, e(x, n))}.
\end{aligned}$$

In the proof of Theorem 23 we have seen that

$$\mathbb{P}_y\left(\tau_{x,r} < \frac{n}{2}\right) \geq c \exp -Cl_9\left(x, y, \frac{n}{2}, d-r\right),$$

which finally yields that

$$\begin{aligned}
\tilde{p}_n(y, x) &\geq \frac{c}{V(x, e(x, n))} \exp -Cl_9\left(x, y, \frac{n}{2}, d-r\right) \\
&\geq \frac{c}{V(x, e(x, n))} \exp -Cl_9\left(x, y, \frac{1}{2}n, d\right).
\end{aligned}$$

■

5 Heat kernel lower bound for graphs

In this section the following off-diagonal lower bound is proved.

Theorem 36 *Let us assume that the graph (Γ, μ) satisfies (p_0) . We also suppose that (\overline{E}) and the elliptic Harnack inequality (H) hold. Then there are $c, C, D > 0$ constants such that for all $x, y \in \Gamma, n \geq d(x, y)$*

$$\tilde{p}_n(x, y) \geq \frac{c}{V(x, e(x, n)) r^D} \exp\left(-Cl_9\left(x, y, \frac{n}{2}\right)\right).$$

where $e(x, n)$ is the inverse of $E(x, R)$ in the second variable and $l = l_9\left(x, y, \frac{n}{2}\right)$, $d = d(x, y)$, $r = \frac{d}{3l}$.

Corollary 37 *If we assume in addition to the conditions of Theorem 36 that $\beta' > 1$ in (13) then the following more readable estimate holds:*

$$\tilde{p}_n(x, y) \geq \frac{c}{V(x, e(x, n)) r^D} \exp\left(-C \left[\frac{E(x, d)}{n}\right]^{\frac{1}{\beta'-1}}\right).$$

This corollary is an easy consequence of Theorem 36.

Remark 38 Let us rephrase the statement of Theorem 36 and Corollary 37. Denote $l = l_9(x, y, \frac{n}{2})$. The trivial recalculation of the estimate

$$\begin{aligned}\tilde{p}_n(x, y) &\geq \frac{c}{V(x, e(x, n)) r^D} \exp(-Cl) \\ &= \frac{c}{V(x, e(x, n)) d(x, y)^D} \exp(D \log 3l - Cl) \\ &\geq \frac{c}{V(x, e(x, n)) d(x, y)^D} \exp(-Cl)\end{aligned}$$

clearly shows the difference between the classical lower bound and the present one. If (8) and (13) hold with $\beta' > 1$ furthermore $n < c \frac{d^\beta}{(\log E(x, d))^{\beta-1}}$ then the extra factor $d^D(x, y)$ is absorbed by the exponent:

$$\tilde{p}_n(x, y) \geq \frac{c}{V(x, e(x, n))} \exp\left(-C \left[\frac{E(x, d)}{n}\right]^{\frac{1}{\beta'-1}}\right).$$

Proposition 39 Let us assume that $(p_0), (\overline{E})$ and the elliptic Harnack inequality (H) holds. Then there are $D, c > 0$ such that for $x, y \in \Gamma$, $r = d(x, y)$, $m > CE(x, r)$ the inequality

$$\tilde{p}_m(y, x) \geq \frac{c}{V(x, e(x, m))} r^{-D}$$

holds.

Proof. The proof is based on a modified version of the chaining argument used in the proof of Lemma 28. From Proposition 35 we know that (\overline{E}) implies

$$\tilde{p}_n(x, x) \geq \frac{c}{V(x, e(x, n))} \quad (21)$$

and (11) (see Remark 11). Let us recall (11) and set $A = \max\{9, A_T\}$, $K = \lceil \frac{A}{4} \rceil$. Consider a sequence of times $m_i = \frac{m}{2^i}$ and radii $r_i = \frac{r}{A^i}$. From the condition $m > CE(x, r)$ and (11) it follows that for all i

$$m_i > CE(x, r_i) \quad (22)$$

holds as well. Let us denote $B_i = B(x, r_i)$, $\tau_i = \tau_{B_i}$ and start a chaining.

$$\begin{aligned}\tilde{p}_m(y, x) &= \sum_{k=1}^m \mathbb{P}_y(\tau_1 = k) \min_{w \in \partial B_1} \tilde{p}_{m-k}(w, x) \\ &\geq \sum_{i=1}^{m/2} \mathbb{P}_y(\tau_1 = k) \min_{w \in \partial B_1} \tilde{p}_{m-k}(w, x) \\ &\geq \mathbb{P}_y(\tau_1 < m/2) \min_{1 \leq k \leq m/2} \min_{w \in \partial B_1} \tilde{p}_{m-k}(w, x).\end{aligned}$$

Let us continue in the same way for all $i \leq L := \lceil \log_A r \rceil$.

It is clear that $B_L = \{x\}$ which concludes to

$$\begin{aligned} \tilde{p}_m(y, x) &\geq \min_{w_i \in \partial B_i} P_y(\tau_1 < m/2) \dots \\ &\dots P_{w_j}\left(\tau_j < \frac{m}{2^j}\right) \dots P_{w_L}\left(\tau_L < \frac{m}{2^L}\right) \min_{0 \leq k \leq m-L} \tilde{p}_k(x, x). \end{aligned}$$

From the initial conditions and (11) we have (22) for all j .

Since in the consecutive steps $d(w_i, x) > 4r_{i+1}$ we insert $K - 1$ copies of balls of radius r_{i+1} splitting the distance into equal smaller ones. We do chaining along them prescribing that the consecutive balls are reached in less than m_i/K time. We can choose C so that the conditions of Proposition 27 are satisfied which yields

$$\mathbb{P}_w\left(\tau_i < \frac{m}{2^i}\right) > c_0^K$$

for all $w_j \in B(x, r_j)$ and j . Consequently, using (21) one has

$$\tilde{p}_m(y, x) \geq \frac{c}{V(x, e(x, m))} c_2^L \geq \frac{c}{V(x, e(x, m))} r^{-D}$$

where $D = \frac{\log \frac{1}{c_2}}{\log A}$. ■

Proof of Theorem 36. The proof is a combination of two chaining arguments. First let us use Theorem 23 to reach the boundary of $B(x, r)$, where $r = \frac{d(x, y)}{3l-1}$, $l = l_9(x, y, \frac{n}{2}, d(x, y))$, then we use Proposition 39. ■

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